Periodic synchronization in a driven system of coupled oscillators

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We study dynamic responses of a set of globally coupled oscillators with randomly distributed frequencies, which is, in the absence of external driving, known to exhibit a transition between the incoherent state and the coherent one with spontaneous synchronization. When each oscillator is driven by periodic force, it displays the characteristic mode locking known as Shapiro steps. Under periodic driving of randomly distributed strengths, the system as a whole is shown to exhibit periodic synchronization as well as transitions between the coherent and the incoherent states. The detailed behavior depends on the characteristic strength of driving relative to the driving frequency.

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I. INTRODUCTION

One of the remarkable features of various oscillatory systems in physics, chemistry, and biology is the emergence of coherent motion among their constituents, which is called "collective synchronization" [1–7]. Such self-organizing systems have been conveniently modeled by sets of coupled nonlinear oscillators [1,8–11], the simple class of which includes the globally coupled oscillators. Global coupling, which couples each oscillator to all the others in the system, appears naturally in biological and some physical systems [11]; in other systems with local coupling, it may be regarded as the mean-field approximation. The ideal system in which all the (globally coupled) oscillators are identical is known to be integrable [12]. This is in contrast with the system of oscillators with randomly distributed frequencies, which resembles more closely real systems in nature. It was shown in such a system that collective synchronization indeed sets in gradually as the (global) coupling strength is increased beyond the critical value, which is reminiscent of the second-order phase transition in equilibrium [13].

In many cases, the oscillatory systems to be modeled are not isolated and are subject to the external driving, often periodic in time. At the simplest level, for example, many biological systems are driven by the periodic cycles of the planetary motion, say, daily and yearly, while in the collective behavior of a society, one may consider the role of external perturbations such as election, economic crash, and war. In some cases, external periodic driving such as microwaves [3], alternating currents [4], laser beams [5], repetitive stimuli [7], etc., is applied explicitly to the system. When a single oscillator is driven by periodic force, it is well known to display the characteristic mode locking, which is called the Shapiro steps particularly in the case of a Josephson junction [14]. In a two-dimensional array of *identical* Josephson junctions, coupled locally and driven by alternating currents, coherence is also manifested by the so-called giant Shapiro steps [4,15]. On the other hand, the role of such external driving in a set of globally coupled oscillators with randomly distributed natural frequencies has not been addressed, in particular, with regard to the collective synchronization. For example, one may ask how the driving changes the onset of the collective synchronization and/or the nature of the nonequilibrium transition between the incoherent state and the coherent one.

This work investigates the dynamic response of a set of globally coupled oscillators to the periodic external driving, with emphasis on the effects on the collective synchronization. The results of our investigation can be summarized as follows: When all the oscillators are driven identically, the response of the whole system is rather trivial, and does not affect the magnitude of synchronization. This is to be contrasted with the case that each oscillator is driven by periodic force of different strength. Under such periodic driving, each oscillator still displays the appropriate Shapiro steps. Here, only oscillators locked to the external driving contribute to the collective synchronization of the whole system, which, for driving with randomly distributed strengths, turns out to be periodic in time. The detailed behavior of such *periodic synchronization* depends on the characteristic strength of driving relative to its frequency as well as the coupling strength and the distribution of frequencies. In particular, the system under appropriate conditions can be driven to display periodic synchronization-desynchronization transitions, each of which can be either continuous or discontinuous. Such periodic synchronization might have been already observed in oscillatory responses of neurons in a visual cortex [7].

This paper consists of five sections. Section II introduces the system of driven oscillators with global sinusoidal coupling, beginning with the simplest system: two coupled oscillators under external periodic driving. The two-oscillator system corresponds to a single Josephson junction with an applied current, and for strong coupling, displays phase locking or synchronization when driven identically. The absence of locking in the system with weak coupling simply reflects the current-voltage characteristic in the corresponding Josephson junction. On the other hand, driving each oscillator with different strength corresponds to applying an alternating current to the Josephson junction, which leads to the well-known Shapiro steps. We then present an N-oscillator system, which can also be reduced to a single-oscillator system by introducing a complex order parameter. While the case of uniform driving is rather trivial, driving with nonuniform strength again yields mode locking and characteristic Shapiro steps, depending on the order parameter. Here, the order parameter, which describes the collective synchronization of the system, is itself determined by such locking. Thus, self-consistency is required (see Sec. III). Noting that only locked oscillators contribute to the synchronization, we derive the self-consistency equation for the order parameter, which displays a variety of bifurcation phenomena. In particular, the system is shown to undergo a continuous or discontinuous transition between the incoherent and the coherent states as the coupling strength is varied. In Sec. IV simple examples in which the system is driven with simple strength distribution and with relatively high frequency are investigated in detail to show periodic synchronization explicitly. Even in those simple cases, the system exhibits rich behavior, e.g., several continuous and discontinuous transitions occurring periodically. Strong and low-frequency driving tends to produce more complex oscillations. Finally, Sec. V gives a brief discussion as well as a summary of the main results. Their possible relevance to the observed responses of cortical neurons is also discussed.

II. DRIVEN SYSTEM OF COUPLED OSCILLATORS

Consider a set of N oscillators, the ith of which is described by its phase ϕ_i ($i=1,2,\ldots,N$). The dynamics of the system under periodic driving is governed by the coupled first-order differential equations

$$\dot{\phi}_i=\omega_i+I_i\cos\Omega t-\frac{K}{N}\sum_{j=1}^{N}\sin(\phi_i-\phi_j)\ , \tag{1}$$

where ω_i on the right-hand side is the natural frequency of the ith oscillator, the second term denotes the external driving, and the last term represents the global coupling between the oscillators, with strength K/N. The natural frequency ω_i is distributed over all of the oscillators according to the distribution $g(\omega)$, which is assumed to be smooth and symmetric about ω_0. Without loss of generality, we may take ω_0 to be zero, and we also assume that $g(\omega)$ is concave at $\omega=0$, i.e., $g''(0)<0$. The frequency Ω of the driving is assumed to be uniform for all oscillators, while the strength I_i may vary for different oscillators. In the absence of the driving term, Eq. (1) describes the standard system of coupled oscillators, which has been studied extensively [1,13]: When the interactions are negligible, each oscillator evolves with its own natural frequency, leading to the incoherent state; interactions, on the other hand, tend to lock oscillators and to develop coherent motion. Thus the competition between the two terms determines whether the system will evolve toward collective synchronization. In Eq. (1) the external driving described by the second term provides an additional source for the competition, which is expected to play a crucial role in the synchronization of the system.

To gain some insight into the effects of driving, we first consider the simple case of two coupled oscillators ($N=2$), which is described by the two coupled equations

$$\begin{aligned}\dot{\phi}_1&=\omega_1+I_1\cos\Omega t-\frac{K}{2}\sin(\phi_1-\phi_2)\ ,\\ \dot{\phi}_2&=\omega_2+I_2\cos\Omega t-\frac{K}{2}\sin(\phi_2-\phi_1)\ .\end{aligned} \tag{2}$$

The above two equations can be trivially decoupled by defining the relative phase $\phi\equiv\phi_1-\phi_2$. The equation of motion for ϕ reads

$$\dot{\phi}+K\sin\phi=\omega+I\cos\Omega t\ , \tag{3}$$

where $\omega\equiv\omega_1-\omega_2$ denotes the relative (natural) frequency and $I\equiv I_1-I_2$ represents the difference in the driving strength. We now consider two cases.

First, when the two oscillators are driven by the same driving ($I_1=I_2$), Eq. (3) becomes time independent and easy to analyze: It corresponds to a pendulum with a constant torque or a single resistively shunted Josephson junction with an applied direct (dc) current ω (in reduced units). Here two types of solutions exist depending on the coupling strength, as follows.

(i) $\omega\le K$. In this case the coupling is strong enough to drive the system to the fixed point given by $\phi=\sin^{-1}(\omega/K)$. Thus we have the stationary solution, where the two oscillators are phase locked to each other due to the coupling

$$\begin{aligned}\phi_1&=\bar{\omega}t+\frac{I_1}{\Omega}\sin\Omega t+\phi_0\ ,\\ \phi_2&=\phi_1-\sin^{-1}\frac{\omega}{K}\ ,\end{aligned}$$

with the mean frequency $\bar{\omega}\equiv(\omega_1+\omega_2)/2$ and an arbitrary constant ϕ_0.

(ii) $\omega>K$. Here the coupling loses in the competition with the natural frequency, and the system does not possess a fixed point: ϕ increases continuously, and the two oscillators are not phase locked to each other. The average rate of increase is simply given by $\langle\dot{\phi}\rangle=\sqrt{\omega^2-K^2}$, leading to the (dc) current-voltage characteristic in the case of a Josephson junction [16]. Thus, in general $\langle\dot{\phi}\rangle/\Omega$ is irrational, and the system is not locked to the external driving, either.

Second, when each oscillator is driven with different strength ($I_1\neq I_2$), $I\cos\Omega t$ in Eq. (3) corresponds to an alternating current applied to the junction. Thus we have a resistively shunted Josephson junction driven by a combined direct and alternating current. It is well known that such a system can be locked to the external driving, which is characterized by the Shapiro steps [14]

$$\frac{\langle\dot{\phi}\rangle}{\Omega}=n\ , \tag{4}$$

with n integer. On the nth step, the (locked) phase of the oscillator is given by

$$\phi \approx n\Omega t + \frac{I}{\Omega}\sin\Omega t + \phi_0 , \quad (5)$$

where higher harmonics have been disregarded [17]. Note here that Eq. (5) does not depend on the coupling K explicitly. The coupling strength determines the constant ϕ_0 according to

$$\omega = n\Omega + (-1)^n K J_n(I/\Omega)\sin\phi_0 ,$$

where J_n is the nth Bessel function. Thus the half-width of the step is given by $|J_n(I/\Omega)|$, as is well known. Off the step, the oscillator is unlocked and its phase is, again with higher harmonics disregarded,

$$\phi \approx \omega t + \frac{I}{\Omega}\sin\Omega t + \phi^0 , \quad (6)$$

where ϕ^0 is a constant independent of ω.

We now return to the set of N oscillators described by Eq. (1). Collective synchronization of such an N-oscillator system is conveniently described by the complex *order parameter* [13]

$$\Psi \equiv \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j} = \Delta e^{i\theta} , \quad (7)$$

where nonvanishing Ψ indicates the appearance of synchronization. In the absence of driving, the order parameter is time independent; in the driven system, on the other hand, the magnitude Δ and the phase θ of the order parameter may depend on time.

We first consider the case of uniform driving, not only in frequency but also in strength ($I_i = I$), and define $\tilde{\phi}_i \equiv \phi_i - (I/\Omega)\sin\Omega t$. In terms of the new variables, Eq. (1) takes the form

$$\dot{\tilde{\phi}}_i = \omega_i - \frac{K}{N}\sum_{j=1}^{N}\sin(\tilde{\phi}_i - \tilde{\phi}_j) , \quad (8)$$

where the driving term has been removed. Thus we have the canonical system of coupled oscillators [13,18], for which the appropriate order parameter $\tilde{\Psi} \equiv (1/N)\sum_j \exp(i\tilde{\phi}_j)$ is known to vanish if the coupling is weak, $K < K_c \equiv 2/\pi g(0)$. (For a detailed discussion on the stability of this null solution, see Ref. [18].) As the coupling increases beyond K_c, this null solution becomes unstable and a nonvanishing solution $\tilde{\Psi} = \tilde{\Delta}e^{i\tilde{\theta}}$ appears, signaling synchronization of the system. Here, the magnitude $\tilde{\Delta}$ grows as $(K-K_c)^{\beta}$ with exponent $\beta = \frac{1}{2}$, which coincides with the mean-field value in equilibrium phase transitions. The phase $\tilde{\theta}$ is an arbitrary constant, reflecting the U(1) symmetry of the system. The order parameter Ψ in terms of the original variables is simply given by

$$\Psi = \exp\left[i\frac{I}{\Omega}\sin\Omega t \right]\tilde{\Psi} = \tilde{\Delta}\sum_n J_n(I/\Omega)e^{i(n\Omega t + \theta)} , \quad (9)$$

which has harmonic components rotating on the complex plane.

We next consider the system under driving with nonuniform strength, where I_i is randomly distributed, again symmetrically about zero. The order parameter defined in Eq. (7) still allows us to reduce Eq. (1) into a *single* decoupled equation,

$$\dot{\phi}_i = \omega_i + I_i\cos\Omega t - K\Delta\sin(\phi_i - \theta) ,$$

where Δ and θ are to be determined by imposing self-consistency. We then seek the stationary solution with constant θ, which is possible due to the symmetry of the distributions of ω_i and I_i about zero. Redefining $\phi_i - \theta$ as ϕ_i and suppressing indices for simplicity, we obtain

$$\dot{\phi} + K\Delta\sin\phi = \omega + I\cos\Omega t , \quad (10)$$

which is essentially the same as Eq. (3). Thus, those oscillators of frequency in the range

$$n\Omega - K\Delta|J_n(I/\Omega)| \le \omega \le n\Omega + K\Delta|J_n(I/\Omega)| , \quad (11)$$

with integer n, which will be denoted by the notation $\omega \in S_n$, are locked to the external driving, and their phases in the stationary state are described by Eq. (5), with ϕ_0 given by

$$\phi_0 = (-1)^n \sin^{-1}\left[\frac{\omega - n\Omega}{K\Delta J_n(I/\Omega)} \right] . \quad (12)$$

Oscillators of frequency outside the intervals given by Eq. (11) ($\omega \notin S_n$ for any integer n) are unlocked and are described by Eq. (6).

III. SELF-CONSISTENCY EQUATION FOR THE ORDER PARAMETER

In this section we compute the order parameter, for which self-consistency is imposed. We thus derive the equation for the order parameter, which determines the collective behavior of the system. Suppose that the driving strength I is distributed according to $f(I)$, independently of the frequency ω. Recalling that ϕ in Eq. (10) in fact represent $\phi - \theta$, we have the self-consistency equation

$$\Delta = \frac{1}{N}\sum_j e^{i\phi_j} = \int_{-\infty}^{\infty} dI\, f(I)\int_{-\infty}^{\infty} d\omega\, g(\omega)\langle e^{i\phi}\rangle_{\omega,I} , \quad (13)$$

where $\langle \cdots \rangle_{\omega,I}$ denotes the average in the stationary state with ω and I given. For locked oscillators of frequency $\omega \in S_n$, Eqs. (5) and (12) lead straightforwardly to

$$\langle e^{i\phi}\rangle_{\omega,I} = \exp\left[i\left[n\Omega t + \frac{I}{\Omega}\sin\Omega t \right] \right] \times [\sqrt{1-x^2} + i(-1)^n x] ,$$

where

$$x \equiv (\omega - n\Omega)/K\Delta J_n(I/\Omega) .$$

Unlocked oscillators, on the other hand, have their stationary-state phases in uniform distribution. [Recall that Eq. (6) is essentially the solution of Eq. (3) without coupling ($K=0$). It is easy to see that the corresponding Fokker-Planck equation [19] has the uniform distribution of the phase as its stationary solution.] Therefore, in general they do not contribute to the collective synchronization of the system. Equation (13) thus becomes

$$\Delta=K\Delta\exp\left[i\frac{I}{\Omega}\sin\Omega t\right]\times\sum_n\exp[in\Omega t]\int_{-\infty}^{\infty}dI\,f(I)J_n(I/\Omega)\times\int_{-1}^{1}dx\,g[n\Omega+K\Delta J_n(I/\Omega)x]\times[\sqrt{1-x^2}+i(-1)^n x]\ ,$$

which consists of contributions from all locked oscillators. Assuming $K\Delta\ll 1$ near the transition to the coherent state, we expand $g[n\Omega+K\Delta J_n(I/\Omega)x]$ around $n\Omega$, make use of the symmetry of $g(\omega)$ and $f(I)$, and obtain

$$\Delta=aK\Delta-b(K\Delta)^2-c(K\Delta)^3+O(K\Delta)^4\ , \quad (14)$$

with

$$a\equiv\frac{\pi}{2}g(0)\langle J_0(I)\cos(I\sin t)\rangle+\pi\sum_{n=1}^{\infty}\{g(2n\Omega)\cos 2nt\langle J_{2n}(I)\cos(I\sin t)\rangle-g[(2n-1)\Omega]\sin(2n-1)t\times\langle J_{2n-1}(I)\sin(I\sin t)\rangle\}\ ,$$

$$b\equiv\frac{4}{3}\sum_{n=1}^{\infty}(-1)^n g'(n\Omega)\sin nt\langle J_n^2(I)\cos(I\sin t)\rangle\ , \quad (15)$$

$$c\equiv-\frac{\pi}{16}g''(0)\langle J_0^3(I)\cos(I\sin t)\rangle-\frac{\pi}{8}\sum_{n=1}^{\infty}\{g''(2n\Omega)\cos 2nt\langle J_{2n}^3(I)\cos(I\sin t)\rangle-g''[(2n-1)\Omega]\sin(2n-1)t\times\langle J_{2n-1}^3(I)\sin(I\sin t)\rangle\}\ ,$$

where $\langle X\rangle\equiv\int dI\,f(I)X$ and, for convenience, I and t have been rescaled in units of Ω and Ω^{-1}, respectively.

It is obvious that Eq. (14) has all coefficients real, allowing real solutions for Δ. Since the coefficients given by Eq. (15) are periodic in time, Δ should be also periodic in time. Thus we indeed have the order parameter in the form $\Psi=\Delta e^{i\theta}$ with time-dependent Δ and arbitrary constant θ, which we have been seeking. As a check, we may also consider the system without driving, where $f(I)=\delta(I)$. In this trivial case the coefficients given by Eq. (15) simply become

$$a=\frac{\pi}{2}g(0)\ ,\quad b=0\ ,\quad c=-\frac{\pi}{16}g''(0)\ , \quad (16)$$

and, as expected, Eq. (14) reduces precisely to the self-consistency equation obtained in Ref. [13].

The collective response of the system for given values of a, b, and c (at a given time) can in principle be obtained by solving Eq. (14), which yields, in addition to the null solution $\Delta=0$, nontrivial solutions:

$$\Delta=\Delta_\pm\equiv\frac{-bK\pm\sqrt{(b^2+4ca)K^2-4cK}}{2cK^2}\ , \quad (17)$$

if

$$K\ge K_0\equiv 4c/(b^2+4ca)\ .$$

For simplicity, we assume that both a and c are greater than zero, while b can be of either sign.

When $b=0$, only the null solution is possible for $K<K_c\equiv a^{-1}$, as shown in Fig. 1(a). As K increases beyond K_c, however, the null solution becomes unstable and the nontrivial solution Δ_+ appears, as displayed in Fig. 1(c). Figure 2(a) shows the emergence of the nontrivial solution via a pitchfork bifurcation [20]. It subsequently grows in a continuous manner as $(a^2/\sqrt{c})(K-K_c)^{1/2}$. For $b>0$, the null solution is still the physical solution for $K<K_c$; it loses its stability to the nontrivial solution Δ_+ via a transcritical bifurcation [20] at $K=K_c$. Near K_c, the latter, which originates from the unphysical (negative) solution generated via a tangent bifurcation [20] at $K=K_0$, grows linearly as $(a^3/b)(K-K_c)$ [see Figs. 1 and 2(b)]. Thus, the system for $b\ge 0$ may be regarded as displaying a second-order phase transition into the coherent state as the coupling is increased beyond K_c.

The case $b<0$ is even more interesting, as shown in Figs. 3 and 4: For $K<K_0$, only the null solution exists, while the stable nontrivial solution Δ_+ (together with the unstable solution Δ_-) appears via a tangent bifurcation at $K=K_0$. The null solution then loses its stability to Δ_- via a transcritical bifurcation at $K=K_c$. Thus only the nontrivial solution Δ_+ is stable for $K>K_c$ (the other solution Δ_- becomes negative). For $K_0<K<K_c$, on the other hand, both the null solution and the nontrivial solution Δ_+ are possible, with different basins of attraction. In this bistability region, as K approaches K_0 (from above), the nontrivial solution takes the form

$$\Delta_+\approx\frac{|b|(b^2+4ca)}{8c^2}+\frac{(b^2+4ca)^2}{32c^3}(K-K_0)^{1/2}\ ,$$

while, as K approaches K_c, it becomes

$$\Delta_+\approx\frac{a|b|}{c}+\frac{a^3}{|b|}(K-K_c)\ .$$

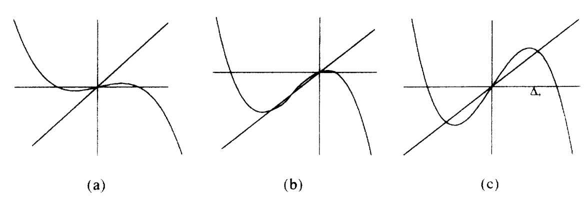

FIG. 1. Graphical solution of Eq. (13) for $b\ge 0$, with (a) $K<K_0$, (b) $K_0<K<K_c$, and (c) $K>K_c$. The negative solutions appearing in (b) and (c) are not physical. Note also that K_0 coincides with K_c for $b=0$.

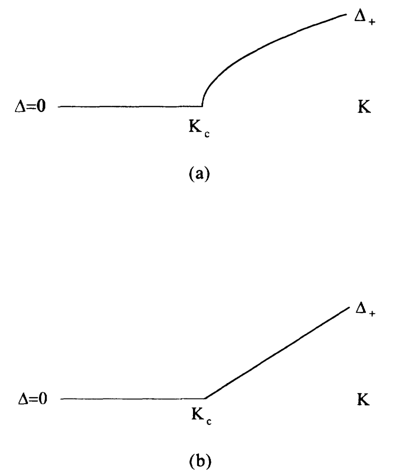

FIG. 2. Bifurcation diagram for (a) $b=0$ and (b) $b>0$, where the solid and dashed lines represent stable and unstable solutions, respectively. (a) shows a pitchfork bifurcation at $K=K_c$, while (b) describes a transcritical bifurcation. The tangent bifurcation at K_0 producing a pair of unphysical (negative) solutions is not drawn in (b).

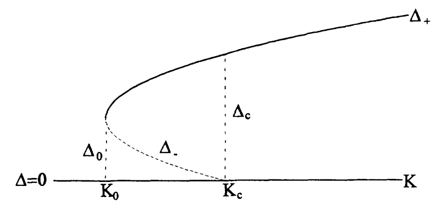

FIG. 4. Bifurcation diagram for $b<0$. A pair of the stable solution Δ_+ and the unstable one Δ_- appears via the tangent bifurcation at K_0; the latter exchanges stability with the null solution via the transcritical bifurcation at K_c, and becomes unphysical. The dotted lines indicate the jumps in the order parameter.

As K is increased from zero, the system, which starts in the incoherent state, remains incoherent even if K grows beyond K_0. As K is increased further to K_c, however, the basin of attraction for the null solution shrinks to zero, driving the system into the coherent state described by the nontrivial solution. Thus, the system exhibits a first-order phase transition at $K=K_c$ into the coherent state, with the jump $\Delta_c \equiv |b|/cK_c$ in the order parameter Δ. Conversely, when K is decreased below K_c, the system is still in the basin of attraction for the nontrivial solution, which shrinks to zero if K is decreased further to K_0. Accordingly, the transition into the incoherent state occurs at $K=K_0$ with the jump $\Delta_0 \equiv |b|/2cK_0$, and in this manner the system displays hysteresis as the coupling strength is varied. Here, the expressions for the jumps Δ_0 and Δ_c in the order parameter are accurate only if $|b| \ll c$; otherwise, the assumption $K\Delta \ll 1$ is not valid, and higher-order terms become relevant. It has been further assumed that $b^2 < 2ca$.

Other cases in which a and/or c are negative can also be investigated. The general features are similar, and will not be discussed here. One additional feature is that there exist ranges of parameters in which neither the null solution nor Δ_+ are stable. This indicates that the present analysis which considers up to the cubic term is insufficient and higher-order terms should be retained; they in general yield a nontrivial stable solution $\tilde{\Delta}$.

IV. PERIODIC SYNCHRONIZATION

In our system the coupling K is in general fixed and uncontrollable. Instead, K_0 and K_c change with time due to the (periodic) time dependence of a, b, and c, the explicit form of which can be obtained by Eq. (15) for given distributions $g(\omega)$ and $f(I)$. In this section we suppose that the oscillators have their natural frequencies distributed in the interval $[-\omega_1,\omega_1]$, i.e., $g(\omega)\neq 0$ only for $|\omega|<\omega_1$, and examine simple cases to show periodic synchronization explicitly.

We first consider higher-frequency driving such that $\Omega > \omega_1$. This is the simplest driven system, in which locking is possible only for $n=0$. Equation (15) then yields

$$a=(\pi/2)g(0)\langle J_0(I\sin t)\rangle\ ,$$

$$b=0\ ,$$

and

$$c=-(\pi/16)g''(0)\langle J_0^3(I)\cos(I\sin t)\rangle\ .$$

To proceed further, we need the explicit form of the distribution $f(I)$, and for simplicity we will mainly consider a $\pm I$-type distribution,

$$f(I)=\tfrac{1}{2}[\delta(I-I_0)+\delta(I+I_0)]\ . \tag{18}$$

A broad distribution, e.g., a Gaussian, can also be considered, but the characteristic features such as periodic synchronization are largely similar. With the above $\pm I$ distribution, we have

$$\begin{aligned} a&=\frac{\pi}{2}g(0)J_0(I_0)\cos(I_0\sin t)\ ,\\ b&=0\ ,\\ c&=-\frac{\pi}{16}g''(0)J_0^3(I_0)\cos(I_0\sin t)\ , \end{aligned} \tag{19}$$

and the collective response of the system depends on I_0, the driving strength relative to the driving frequency. (Recall that I_0 has been defined in units of Ω. In the case

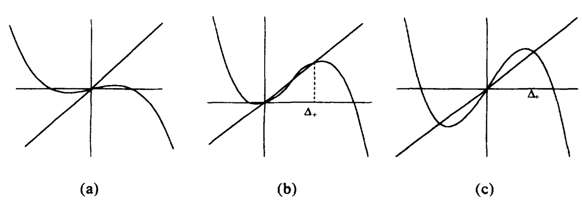

FIG. 3. Graphical solution of Eq. (13) for $b<0$, with (a) $K<K_0$, (b) $K_0<K<K_c$, and (c) $K>K_c$. In (b), both the null solution and the nontrivial solution Δ_+ are stable.

of a Gaussian distribution, I_0 corresponds to the root-mean-square value of the driving strength.) For weak driving ($I_0 < \pi/2$), a as well as c is always greater than zero, reaching its maximum

$$a_{\max} \equiv (\pi/2) g(0) J_0(I_0)$$

and minimum

$$a_{\min} \equiv (\pi/2) g(0) J_0(I_0) \cos I_0$$

periodically. Since the nontrivial solution exists only for $K > K_c$ ($\equiv a^{-1}$), we find that for $K < K_c^{\min} \equiv a_{\max}^{-1}$, the system is always incoherent ($\Delta = 0$), while it is always in the coherent state, displaying nonzero synchronization ($\Delta = \Delta_+$), for $K > K_c^{\max} \equiv a_{\min}^{-1}$. Note that Δ_+ itself is periodic in time due to the periodicity of a and b. Figure 5(a) shows the periodic synchronization of period π/Ω (recall that t has been defined in units of Ω^{-1}) displayed by the system in this case. For intermediate couplings ($K_c^{\min} < K < K_c^{\max}$), we have $K < K_c$ for periodic intervals of time, while $K > K_c$ for the rest of the time, and the system oscillates between the two states: $\Delta = 0$ for

$$n\pi + t_c < t < (n+1)\pi - t_c$$

and $\Delta = \Delta_+$ for $n\pi - t_c < t < n\pi + t_c$, where t_c ($\in [0, \pi/2]$) has been defined by the relation $K_c(t_c) = K$ or $a(t_c) K = 1$. Here, it is simply given by

$$t_c = \sin^{-1}\{ I_0^{-1} \cos^{-1}[2/\pi g(0) K J_0(I_0)] \} .$$

Again, Δ_+ is periodic, and in particular, vanishes at time $t = n\pi \pm t_c$. Therefore, the order parameter Δ is a continuous (periodic) function of time, and the system exhibits continuous transitions periodically between the coherent and incoherent states, as shown schematically in Fig. 5(b). When the driving gets stronger such that $\pi/2 < I_0 < 3\pi/2$, the general behavior of the system does not change for $K < K_c^{\max}$, i.e., Δ is always zero for $K < K_c^{\min}$, while it oscillates between zero and Δ_+ for $K_c^{\min} < K < K_c^{\max}$. For $K > K_c^{\max}$, on the other hand, the system displays oscillation whose cycle consists of two coherent states separated by two incoherent states. It then undergoes four synchronization-desynchronization transitions in a cycle. Even stronger driving leads to more complex oscillations in the synchronization of the system.

We next consider driving with lower frequency such that $\Omega < \omega_1 < 2\Omega$, where locking is possible for $n = 1$ in addition to $n = 0$. For the $\pm I$ distribution in Eq. (18), Eq. (15) reduces to

$$\begin{aligned} a &= \frac{\pi}{2} g(0) J_0(I_0) \cos(I_0 \sin t) \\ &\quad - \pi g(\Omega) J_1(I_0) \sin(I_0 \sin t) \sin t , \\ b &= -\tfrac{4}{3} g'(\Omega) J_1^3(I_0) \cos(I_0 \sin t) \sin t , \\ c &= -\frac{\pi}{16} g''(0) J_0^3(I_0) \cos(I_0 \sin t) \\ &\quad + \frac{\pi}{8} g''(\Omega) J_1^3(I_0) \sin(I_0 \sin t) \sin t , \end{aligned} \tag{20}$$

which lead to a variety of oscillatory behavior (of period $2\pi/\Omega$) depending on the ranges of parameters. For example, in the simplest case of weak driving ($I_0 < \pi/2$) such that

$$a_{\min} \equiv (\pi/2) g(0) J_0(I_0) \cos I_0 - \pi g(\Omega) J_1(I_0) \sin I_0 > 0$$

and $c > 0$, typical responses of the system are as follows: For $K < K_c^{\min}$ ($\equiv a_{\max}^{-1}$), the system remains incoherent ($\Delta = 0$). [Here the expression of $a_{\max} = a(t=0)$ is the same as before.] For $K_c^{\min} < K < K_0^{\max}$, where $K_0^{\max}$ denotes the maximum value of K_0 [$\equiv 4c/(b^2 + 4ca)$, as defined in Sec. III, with a, b, and c given by Eq. (20)], the typical behavior of the order parameter Δ is shown schematically in Fig. 6(a). Since $a(t=0)K > 1$, the order parameter evolves from finite Δ_+ at $t=0$ and decreases

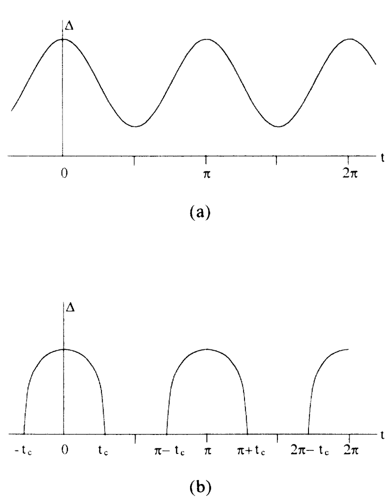

FIG. 5. Schematic diagram of the periodic synchronization for $\Omega > \omega_1$, with (a) $K > K_c^{\max}$ and (b) $K_c^{\min} < K < K_c^{\max}$. (b) exhibits continuous synchronization-desynchronization transitions.

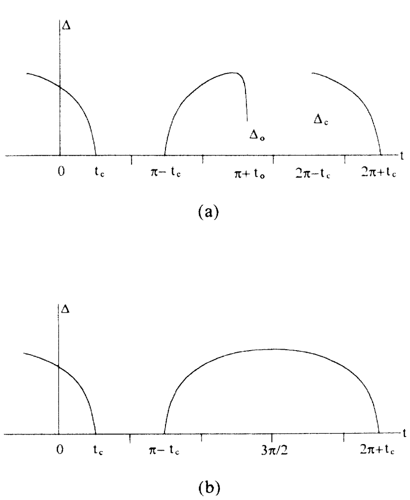

FIG. 6. Schematic diagram of the periodic synchronization for $\Omega < \omega_1 < 2\Omega$, with (a) $K_c^{\min} < K < K_0^{\max}$ and (b) $K_0^{\max} < K < K_c^{\max}$. (a) shows discontinuous as well as continuous transitions, while (b) displays only continuous transitions.

continuously with time, reaching zero at $t=t_c$, where $a(t_c)K=1$. It stays at zero until $t=\pi-t_c$, when it begins to increase, going back to Δ_+. Then it jumps down to zero at $t=\pi+t_0$, where t_0 ($\in[0,\pi/2]$) is determined by the relation $K_0(t_0)=K$, *with discontinuity*

$$\Delta_0(t=\pi+t_0)=b(t_0)/2c(t_0)K \ .$$

The order parameter eventually jumps back to Δ_+ at $t=2\pi-t_c$ again with discontinuity

$$\Delta_c(t=2\pi-t_c)=b(t_c)/c(t_c)K \ ,$$

completing one cycle. Thus, the system displays four synchronization-desynchronization transitions in a cycle. Of the four transitions, two are continuous, while the remaining two are discontinuous. As the coupling is increased further so that $K_0^{\max}<K<K_c^{\max}$, the order parameter oscillates between zero and Δ_+ in a continuous manner, which is displayed in Fig. 6(b). Finally, for $K>K_c^{\max}$, the system always remains coherent ($\Delta=\Delta_+$), displaying simple periodic synchronization. For other ranges of parameters, in particular, for stronger driving, there appear more complex oscillations composed of many transitions per cycle. For example, when I_0 is increased beyond $\pi/2$, the system with strong coupling ($K>K_c^{\max}$) can exhibit eight transitions in a cycle.

V. CONCLUSIONS

We have studied dynamic responses of a set of globally coupled oscillators with randomly distributed natural frequencies, which are conveniently described by introducing a complex order parameter. When all the oscillators are driven identically, the response of the whole system has been shown to be rather trivial, and does not affect the magnitude of synchronization. This is to be contrasted with the case in which each oscillator is driven by periodic force of randomly distributed strength. Under such driving, each oscillator still displays the appropriate Shapiro steps, and only oscillators locked to the external driving contribute to the collective synchronization of the whole system. The self-consistency condition then gives the equation for the order parameter, which in turn leads to the possibility of collective synchronization that is periodic in time. The detailed behavior of such periodic synchronization has been shown to depend on the characteristic driving strength relative to the driving frequency as well as the coupling strength and the distribution of natural frequencies. In particular, strong driving with frequency low compared to the width of the natural frequency distribution tends to yield complex responses such as periodic transitions between the coherent and incoherent states; some of the transitions may be discontinuous, while others are continuous.

This periodic synchronization might be of relevance to the interesting oscillatory responses of cortical neurons reported recently [7], where oscillations of the phase difference between the two "locked" signals have been observed within a single stimulus period. When the two signals have zero phase difference, they are indeed synchronized and contribute to the order parameter defined in Eq. (7). Two signals which are out of phase (i.e., with phase difference π), on the other hand, do not contribute to the order parameter. Correspondingly, the oscillation of the phase difference leads to the periodic variation of the order parameter, which indicates that the system appears to display periodic synchronization.

It should be stressed that the analytical tractability of this study stems from the global coupling, i.e., the mean-field nature of the system, which allows us to reduce the set of N coupled equations into a single equation for the order parameter. A single first-order differential equation in the form of Eq. (10) is well known to produce only integer Shapiro steps given by Eq. (4), and the corresponding locked oscillators in turn determine the collective synchronization of the system. In a system of locally coupled oscillators, on the other hand, such a reduction is in general unattainable except in the ideal system of identical oscillators; the coupled equations describing the latter can also be reduced to a single equation in the form of Eq. (3) by symmetry consideration. Accordingly, a two-dimensional array of identical Josephson junctions, driven by external currents, displays only integer Shapiro steps unless a magnetic field is applied [4,15]. Here, an applied magnetic field introduces "frustration" to the system, allowing reduction into a few coupled equations (again by symmetry) rather than into a single equation. Thus, the arrays of identical junctions in magnetic fields as well as those of dissimilar junctions (without magnetic fields) are described by sets of coupled equations, where different characteristic frequencies coexist, and can exhibit both integer and fractional steps [4,15]. Further, evidence for steps at every rational, which is suggestive of devil's staircase structure, has been reported in such systems [21]. It would then be of great interest to investigate the behavior of collective synchronization in those systems displaying fractional steps and, in particular, devil's staircase structure. Such structure presumably tends to suppress collective synchronization, but apparently leads to the possibility of complex behavior such as chaos in extended systems. Note here that the simplest case of two coupled equations may be expressed as a second-order differential equation, which describes, e.g., a Josephson junction with nonvanishing capacitance. In the appropriate regime, the corresponding Poincaré return map reduces essentially to the circle map, which is indeed known to display rich behavior including a devil's staircase and chaos [22,23]. This might be helpful in understanding the effects of fractional locking on collective synchronization, the detailed investigation of which is left for further study.

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[1] For a list of references, see A. T. Winfree, *The Geometry of Biological Time* (Springer-Verlag, New York, 1980); Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer-Verlag, Berlin, 1984); L. Glass and M. C. Mackey, *From Clocks to Chaos: The Rhythm of Life* (Princeton University Press, Princeton, NJ, 1988).

[2] H. Haken, *Advanced Synergetics* (Springer-Verlag, Berlin, 1983); R. Serra, M. Andretta, M. Compiani, and G. Zanarini, *Introduction to the Physics of Complex Systems* (Pergamon, Oxford, 1986).

[3] T. D. Clark, Phys. Lett. A **27**, 585 (1968); Phys. Rev. B **8**, 137 (1973).

[4] S. P. Benz, M. Rzchowski, M. Tinkham, and C. J. Lobb, Phys. Rev. Lett. **64**, 693 (1990); H. C. Lee *et al.*, Phys. Rev. B **44**, 921 (1991).

[5] J. L. Bobin, Phys. Rep. **122**, 173 (1985).

[6] T. J. Walker, Science **166**, 891 (1969); M. K. McClintock, Nature **229**, 244 (1971); D. C. Michaels, E. P. Maytyas, and J. Jalife, Circ. Res. **61**, 704 (1987); J. Buck, Q. Rev. Biol. **63**, 265 (1988); R. D. Traub, R. Miles, and R. K. S. Wong, Science **243**, 1319 (1989).

[7] R. Eckhorn *et al.*, Biol. Cybern. **60**, 121 (1988); C. M. Gray and W. Singer, Proc. Nat. Acad. Sci. U.S.A. **86**, 1698 (1989); C. M. Gray, A. K. Engel, P. König, and W. Singer, in *Nonlinear Dynamics and Neuronal Networks,* edited by H. G. Schuster (VCH, Weinheim, 1991), pp. 27–55.

[8] D. R. Tilley, Phys. Lett. A **33**, 205 (1970); R. Bonifacio, F. Casagrande, and M. Milani, *ibid.* **101**, 427 (1984); J. S. Chung, K. H. Lee, and D. Stroud, Phys. Rev. B **40**, 6570 (1989); S. Kim and M. Y. Choi, *ibid.* **48**, 322 (1993).

[9] *Nonlinear Dynamics and Neuronal Networks,* edited by H. G. Schuster (VCH, Weinheim, 1991); *Coupled Oscillating Neurons,* edited by J. G. Taylor and C. L. T. Mannion (Springer-Verlag, London, 1992).

[10] H. Sompolinsky, D. Golomb, and D. Kleinfeld, Phys. Rev. A **43**, 6990 (1991); M. Tsodyks, I. Mitkov, and H. Sompolinsky, Phys. Rev. Lett. **71**, 1280 (1993); D. Hansel, G. Mato, and C. Meunier, Europhys. Lett. **23**, 367 (1993); D. Cai, Y.-C. Lai, and R. L. Winslow, Phys. Rev. Lett. **71**, 2501 (1993).

[11] P. Hadley, M. R. Beasley, and K. Wiesenfeld, Phys. Rev. B **38**, 8712 (1988); K. Y. Tsang, R. E. Mirollo, S. H. Strogatz, and K. Wiesenfeld, Physica D **48**, 102 (1991); G. E. James, E. M. Harrell II, and R. Roy, Phys. Rev. A **41**, 2778 (1990); K. Otsuka, Phys. Rev. Lett. **67**, 1090 (1991); J. Levy, M. S. Sherwin, F. F. Abraham, and K. Wiesenfeld, *ibid.* **68**, 2968 (1992).

[12] S. Watanabe and S. H. Strogatz, Phys. Rev. Lett. **70**, 2391 (1993).

[13] Y. Kuramoto, in *Proceedings of the International Symposium on Mathematical Problems in Theoretical Physics,* edited by H. Araki (Springer-Verlag, New York, 1975); Y. Kuramoto and I. Nishikawa, J. Stat. Phys. **49**, 569 (1987).

[14] S. Shapiro, Phys. Rev. Lett. **11**, 80 (1963); S. Shapiro, A. R. Janus, and S. Holly, Rev. Mod. Phys. **36**, 223 (1964).

[15] K. H. Lee, D. Stroud, and J. S. Chung, Phys. Rev. Lett. **64**, 962 (1990); J. U. Free *et al.*, Phys. Rev. B **41**, 7267 (1990); M. Y. Choi. *ibid.* **46**, 564 (1992).

[16] V. Ambegaokar and B. I. Halperin, Phys. Rev. Lett. **22**, 1364 (1969).

[17] To be precise, this corresponds to a Josephson junction with the voltage bias instead of the current bias. However, there is no qualitative difference in general features, and Eq. (4) also describes the behavior of the current biased junction fairly well. Inclusion of the higher harmonics does not alter the subsequent analysis in the lowest order.

[18] S. H. Strogatz and R. E. Mirollo, J. Stat. Phys. **63**, 613 (1991); S. H. Strogatz, R. E. Mirollo, and P. C. Matthews, Phys. Rev. Lett. **68**, 2730 (1992).

[19] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1989).

[20] See, e.g., P. Bergé, Y. Pomeau, and C. Vidal, *Order within Chaos* (Wiley, New York, 1984).

[21] J. Kim and H. J. Lee, Phys. Rev. B **47**, 582 (1993); S. Kim and M. Y. Choi, Europhys. Lett. **23**, 217 (1993).

[22] P. Bak, Rep. Prog. Phys. **45**, 587 (1982). See also H. G. Schuster, *Deterministic Chaos* (Physik-Verlag, Weinheim, 1984); E. A. Jackson, *Perspectives of Nonlinear Dynamics* (Cambridge University Press, Cambridge, England, 1989), and references therein.

[23] The devil's staircase structure related to the circle map has also been found in cardiac rhythms. See H.-X. Wang, R. de Paola, and W. I. Norwood, Phys. Rev. Lett. **70**, 3671 (1993).